RELATIVELY UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS*

BY

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E. H. Moore† has introduced the notion of uniform convergence of a sequence of functions relative to a scale function. It is the purpose of this paper to study this type of convergence in the field of functions of a real variable.

1. The following definition of relatively uniform convergence is, for the case of a sequence of functions of a real variable, equivalent to the definition given by Moore.

A sequence $\{ \mu_n \}$ of single-valued, real-valued functions μ_n of a variable p, ranging over a set $\mathfrak{P}$ of real elements p, converges *relatively uniformly* on $\mathfrak{P}$ in case there exist functions θ and σ, defined on $\mathfrak{P}$, and for every m an integer n_m (dependent on m), such that for every n greater than or equal to n_m the inequality

$$m \,|\, \theta - \mu_n \,| \leqq |\, \sigma \,| \ddagger \tag{1}$$

holds for every element p of $\mathfrak{P}$.

The function σ of the definition above is called a *scale function.* The sequence $\{ \mu_n \}$ is said to converge *uniformly relative to the scale function* σ; or more simply, *relatively uniformly.*

2. The following propositions are immediate consequences of the definition of relatively uniform convergence.§

I. Uniformity of convergence relative to a constant scale function different from zero is equivalent to uniform convergence.

II. Uniformity relative to σ implies uniformity relative to every function τ such that $|\, \tau \,| \overline{\overline{>}} |\, \sigma \,|$.

* Presented to the Society March 22, 1913.

† *An Introduction to a Form of General Analysis, The New Haven Mathematical Colloquium,* Yale University Press, New Haven, 1910. This memoir will be cited as I. G. A.

‡ The relation (1) holds identically in p. In such cases we follow the usage of Moore (cf. I. G. A., p. 27) and omit the variable.

§ Cf. I. G. A., p. 33, et seq. All propositions and theorems of this paper are stated for sequences of functions. The corresponding propositions for series of functions are readily inferred.

III. Uniformity as to a function σ such that $A \leqq |\sigma| \leqq B$, where A and B are positive, implies uniform convergence.

IV. If a sequence converges uniformly relative to a scale function σ, but does not converge uniformly, then σ is not bounded.

V. If a sequence of functions is defined on a class $\mathfrak{P}$ and if $\mathfrak{P}$ may be divided into a finite number of parts such that the sequence converges relatively uniformly on each part, then the sequence converges relatively uniformly on $\mathfrak{P}$.*

VI. If a sequence converges relatively uniformly on $\mathfrak{P}$, $\mathfrak{P}$ may be divided into a sequence $\{ \mathfrak{P}_n \}$ such that no two sets $\mathfrak{P}_{n_1}$, $\mathfrak{P}_{n_2}$ $(n_1 \neq n_2)$ have a common element, and such that on each $\mathfrak{P}_n$ the sequence converges uniformly.

3. The following examples are illustrative of relative uniformity of convergence.

(*a*) The class $\mathfrak{P}$ is the interval $0 \leqq p \leqq 1$; the sequence $\{ \mu_n \}$ such that $\mu_n(p) = 1/np$ $(p \neq 0)$, $\mu_n(0) = 0$. This sequence does not converge uniformly, but the function $\sigma(p) = 1/p$ is effective as a scale function.

(*b*) The class $\mathfrak{P}$ is the infinite segment, $1 \leqq p$; $\mu_n(p) = 1/np$. The sequence converges uniformly, but satisfies the stronger condition of uniform convergence relative to the scale function $\sigma(p) = 1/p$.†

4. Using the notation of § 1, we denote by $\phi_n(p)$ the least upper bound of $|\theta(p) - \mu_{n'}(p)|$ for all $n' \geqq n$. If the sequence $\{ \mu_n \}$ converges to θ uniformly, relative to a scale function $\sigma(p)$, then for every m there exists an n_m such that the inequality (1) is satisfied for all $n \geqq n_m$. Hence we may write, in view of the definition of $\phi_n(p)$,

$$m \phi_{n_m}(p) \leqq |\sigma(p)|. \tag{2}$$

As an immediate consequence of this result we have the following theorem:

THEOREM 1. *A necessary and sufficient condition that a sequence* $\{ \mu_n \}$ *of functions* μ_n *converges relatively uniformly on* $\mathfrak{P}$ *to a limit function* θ *is that there exist a sequence* $\{ n_m \}$ *of positive integers such that the sequence* $m \phi_{n_m}(p)$ *has an upper bound* $B(p)$ *for every* p.

5. Let $\mathfrak{P}$ be any interval (a, b), and (r, q) a sub-interval of $\mathfrak{P}$ in which the only point of non-uniform convergence‡ of a sequence $\{ \mu_n \}$ is the end point q. The sequence $\{ \mu_n \}$ being supposed convergent on (r, q), we have a function $A(p) \geqq 1$ such that for every p in (r, q)

* Cf. § 6 of this paper, proposition VIII, which is an extension of proposition V and the converse of VI.

† Other examples are given later in this paper. Cf. also I. G. A.

‡ A point q is a point of non-uniform convergence in case the measure of non-uniformity of convergence of the sequence is greater than zero at q. Cf. W. H. Young, Proceedings of the London Mathematical Society, ser. 2, vol. I (1903–4) p. 91; also E. W. Hobson, *Theory of Functions of a Real Variable*, Cambridge University Press, p. 474, § 342; p. 484, § 349.

$$|\theta(p) - \mu_n(p)| \leqq A(p).$$

We will now show that the sequence $\{\mu_n\}$ converges uniformly on (r, q) relative to the scale function $\sigma(p) = A(p)/(q-p)$; $\sigma(q) = A(q)$. For each m greater than $1/(q-p)$ choose p_m so that $q - p_m = 1/m$. Then in the interval (r, p_m), which contains no point of non-uniformity, the sequence $\{\mu_n\}$ converges uniformly, and therefore n_m exists so that for every $n \gtreqless n_m$, p in (r, p_m), and $p = q$,

$$m|\theta(p) - \mu_n(p)| \leqq 1.$$

But for every n, and p in the segment $p_m \leqq p < q$,

$$m|\theta(p) - \mu_n(p)| \leqq A(p)/(q-p).$$

The combination of the last two inequalities gives the desired convergence relative to σ.

The result just obtained is stated in the following theorem:

Theorem 2. *If q is on the left an isolated point of non-uniformity of convergence of a convergent sequence of functions, there exists a left neighborhood of q on which the sequence converges relatively uniformly.*

A similar statement holds if q is isolated on the right.

6. We have occasion to use the following two propositions from general analysis.*

VII. If a sequence of functions is defined on an enumerable set $\mathfrak{P}$ and converges on $\mathfrak{P}$ it converges relatively uniformly on $\mathfrak{P}$.

VIII. If a sequence of functions is defined and converges relatively uniformly on each of a sequence of classes $\mathfrak{P}_n$ it converges relatively uniformly on the class $\mathfrak{P}$ of all elements which are in some class $\mathfrak{P}_n$, the least common superclass of the classes $\mathfrak{P}_n$.

7. Denote by $\mathfrak{Q}$ the set of all points q of non-uniform convergene of a sequence of functions converging on an interval $\mathfrak{P}$ to a definite limit function; by $\mathfrak{Q}'$ the derived set of $\mathfrak{Q}$, and by $\mathfrak{Q}^\circ$ the aggregate of the points of $\mathfrak{Q}$ and $\mathfrak{Q}'$. $\mathfrak{Q}^\circ$ is closed. The set complementary to $\mathfrak{Q}^\circ$ consists of the interior points of an enumerable set of non-overlapping intervals, in each of which the sequence of functions converges relatively uniformly (Theorem 2). It follows from proposition VIII that the sequence converges relatively uniformly on the complement of $\mathfrak{Q}^\circ$. Since the points of $\mathfrak{Q}^\circ$ which are not in $\mathfrak{Q}'$ form an enumerable set, we may state, in view of the preceding remarks and propositions V and VII, the following theorem:

Theorem 3. *A necessary and sufficient condition that a sequence of functions converges relatively uniformly on an interval $\mathfrak{P}$ is that it converges relatively uni-*

* Proposition VII is from I. G. A., p. 87. A simple proof may be given following the lines of the proof of Theorem 2 above. Proposition VIII is an immediate extension of VII.

formly on the derived set $\mathfrak{Q}'$ *of the set* $\mathfrak{Q}$ *of the points of* $\mathfrak{P}$ *which are points of non-uniformity.*

COROLLARY I. *If* $\mathfrak{Q}'$ *is enumerable the sequence converges relatively uniformly on* $\mathfrak{P}$.

COROLLARY II. *If a sequence of functions does not converge relatively uniformly on* $\mathfrak{P}$, *the corresponding set* $\mathfrak{Q}'$ *is not enumerable, that is,* $\mathfrak{Q}$ *is dense on a perfect set.*

8. The converse of corollary II above is not true, as is shown by the following example.* $\mathfrak{P}$ is the interval $(0, 1)$. We define the sequence $\{\mu_n\}$ as follows: $\mu_n(p) = 0$, if p is irrational, zero, or equal to m/k (m and k relatively prime, and $k \neq n$); $\mu_n(p) = 1$, if $p = m/n$ (m, n relatively prime). For this sequence of functions every p is a point of non-uniformity of convergence. However, the functions of the sequence are all zero on the irrational points. Hence by propositions V and VII the sequence converges relatively uniformly.

Osgood† has given an example of a sequence of *continuous* functions converging to zero as a limit for which the set $\mathfrak{Q}$ and its derivative form a perfect set, but which converges relatively uniformly because the sequence is identically zero on $\mathfrak{Q}$ and $\mathfrak{Q}'$.

9. For sequences of continuous functions we have the following theorem:

THEOREM 4. *If a sequence of continuous functions converges on an interval* $\mathfrak{P}$ *to a continuous limit in such a way that the set* $\mathfrak{Q}$ *of all points of non-uniformity of convergence is dense on* $\mathfrak{P}$, *the sequence does not converge relatively uniformly.*

To establish this theorem it is sufficient to show that for every sequence $\{n_m\}$ of positive integers n_m there exists a p determined by the sequence $\{n_m\}$ such that the sequence $\{m\phi_{n_m}(p)\}$‡ is not bounded. Theorem 4 follows in view of Theorem 1.

Let $\{n_m\}$ be any sequence of integers with index m, and $\{\mu_n\}$ denote the sequence of continuous functions of the theorem, converging to a continuous limit θ. Since the sequence $\{\mu_n\}$ does not converge uniformly in any subinterval of $\mathfrak{P}$ there exists an integer m_1 with the following property: for every N there exists an n_1 greater than N and a p_1 (interior to $\mathfrak{P}$) such that

$$m_1 |\theta(p_1) - \mu_{n_1}(p_1)| > 1.$$

Since θ and μ_{n_1} are continuous, we may find an interval I_1 interior to $\mathfrak{P}$ such that for every p in I_1

$$m_1 |\theta(p) - \mu_{n_1}(p)| > 1.$$

* Cf. W. H. Young, Proceedings of the London Mathematical Society, Ser. 2, vol. I (1903–4), p. 94; also E. W. Hobson, *Theory of Functions of a Real Variable*, p. 487.

† *Non-Uniform Convergence and the Integration of Series Term by Term*, American Journal of Mathematics, vol. 19 (1897), p. 168.

‡ Cf. § 4 for explanation of this notation.

Consequently, if we take N equal to n_{m_1} of the sequence $\{ n_m \}$, we have

$$m_1 \phi_{n_{m_1}}(p) > 1$$

for every p in I_1.

By similar reasoning we may show that there exists an m_2 and an interval I_2 interior to I_1 such for every p in I_2

$$m_2 \phi_{n_{m_2}}(p) > 2.$$

Proceeding step by step in the manner indicated we may obtain a sequence $\{ m_j \}$ and a sequence $\{ I_j \}$ of closed intervals, each interior to the preceding, such that for every j we have on I_j

$$m_j \phi_{n_{m_j}}(p) > j.$$

There must exist an element p° common to the intervals I_j. For such an element we have for every j

$$m_j \phi_{n_{m_j}}(p^\circ) > j.$$

The sequence $m \phi_{n_m}(p^\circ)$ is therefore unbounded, which was to be proved.

Osgood* has given an example of a sequence of continuous functions converging to a continuous limit for which the set $\mathfrak{Q}$ is dense on $\mathfrak{P}$. It follows from the theorem just proved that this sequence does not converge relatively uniformly.

10. A point of non-uniformity of convergence of a sequence of functions relative to a subset $\mathfrak{R}$ of $\mathfrak{P}$ is a point whose measure of non-uniformity of convergence relative to $\mathfrak{R}$ is greater than zero.†

Evidently a point of non-uniformity of convergence relative to $\mathfrak{R}$ is a point of non-uniformity of convergence relative to any set containing $\mathfrak{R}$, and therefore of $\mathfrak{P}$.

A slight extension of the proof of Theorem 4 serves to establish the following generalization of that theorem.

THEOREM 5. *If a sequence of functions defined on* $\mathfrak{P}$ *and continuous on* $\mathfrak{R}$, *a perfect subset of* $\mathfrak{P}$, *converges on* $\mathfrak{P}$ *to a limit which is continuous on* $\mathfrak{R}$ *in such a fashion that the class* $\mathfrak{Q}$ *of all points of* $\mathfrak{R}$ *which are points of non-uniformity of convergence relative to* $\mathfrak{R}$ *is dense on* $\mathfrak{R}$, *then the sequence of functions does not converge relatively uniformly on* $\mathfrak{R}$ *or on* $\mathfrak{P}$.

URBANA, ILL.,
May 7, 1913.

* Osgood, loc. cit., p. 171.
† Hobson, loc. cit., p. 484, § 349.